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Explanation of the ball bearing motor and exact solutions of the related Maxwell equations

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Abstract. We present a theory which explains the mechanism of the ball bearing motor. The motor is the result of the rotating of a ball bearing when a current is passed through it. The applied torque is calculated following a perturbative description of the electromagnetic structure of the balls. The main effects result from the interaction of the magnetic field of the central axis with the induced currents and magnetic fields developed within the balls.

The various current densities and fields are determined by solving exactly the corresponding Poisson equations, which result from the Maxwell equations. Our predicted values for the total power, the efficiency and the various required constants are in excellent agreement with the experimental results.

1. Introduction

A current passing through a pair of ball bearings generates a torque of electromagnetic origin which results to the rotation of the bearings. A cross section of the ball bearing system is shown in figure 1. At sufficiently large currents the system of bearings acts as a motor although the torque seems to be rather small. The behaviour of this motor,

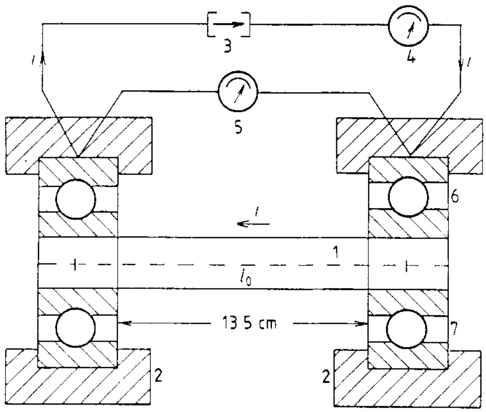

Figure 1. The ball bearing motor (BB): 1 rotating shaft, 2 base of the BB solidly mounted on the laboratory, 3 constant current source, 4 current meter, 5 digital voltometer, 6 spherical ball, 7 bearing.

described for first time in a brief note by Milroy (1967), was studied extensively by Moyssides *et al* 1989 during an experiment using constant current values in the range from 43.5–70.15 A.

For the explanation of this phenomenon, various schemes have been proposed by Gruenberg (1978) and Weenink (1981) unsuccessfully. Gruenberg calculated the torque inside the rotating ball with velocity $\boldsymbol{v}$. He considered the interaction of the primary current of density $\boldsymbol{J}_0$ and its magnetic field $\boldsymbol{B}_0$ with the first-order induced current $\boldsymbol{J}_1$, produced by the force $-e(\boldsymbol{v}\times\boldsymbol{B}_0)$ on each electron e, and its associated magnetic field $\boldsymbol{B}_1$ (figure 2).

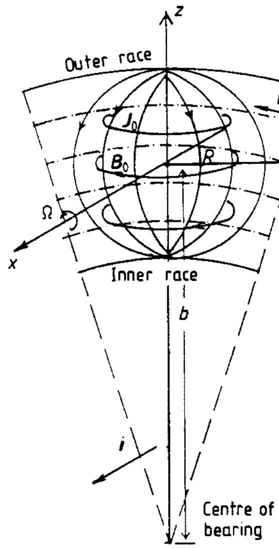

Figure 2. Current and fields of a non-rotating ball: Ω, the angular velocity; $\boldsymbol{J}_0$, primary current density; $\boldsymbol{B}_0$, the magnetic field of the current $\boldsymbol{J}_0$; $\boldsymbol{B}_a$, magnetic field generated by the current i of the central axis; R, radius of the ball.

However, he calculated a non-zero torque, which was found by Weenink to be wrong due to algebraic errors. Weenink, following the same analysis and taking into account the interactions of the previous currents and fields with the second-order induced current, produced by the force $-e(\boldsymbol{v}\times\boldsymbol{B}_1)$ and its associated magnetic field, found a zero torque for the rotating sphere.

From the above schemes, where the current of the central axis and the associated effects have been ignored, we see that the interactions of the lower-order generated currents and associated magnetic fields inside the spherical balls cannot create a torque. This is due to the structure of the various electromagnetic fields within the sphere, independent of the rotational velocity. On the other hand, we estimated that terms of higher order $n\geqslant 2$ are proportional to $(v/c)^n$ where c is the velocity of light. Hence, both of the non-zero contributions to the torque would be of negligible magnitude.

For the understanding of the electromagnetic effects which result in the rotation of the balls and the action of the system of bearings as a motor, we develop the following theory, which takes into account the current of the central axis and the associated effects, and clearly explains the observed behaviour in excellent agreement with experiment.

Let us consider the ball bearing system of figure 2 with the associated coordinate system. A ball is set rotating about the x axis with angular velocity Ω, clockwise (cw) or counterclockwise (ccw) by a push. The motor is not self-starting except when the balls have a residual magnetisation from a previous run. The velocity of every point of the ball defined by the vector $\boldsymbol{r}$ is given by $\boldsymbol{v} = \Omega \boldsymbol{i} \times \boldsymbol{r}$.

A rotating ball is subjected to a torque resulted from the excited $\boldsymbol{J} \times \boldsymbol{B}$ forces, caused by the interactions of the internal primary and induced currents and magnetic fields with those generated by the current of the central axis of the motor, in the vicinity of and inside the ball. A non-rotating ball is subjected to the primary current $\boldsymbol{J}_0$, its magnetic field $\boldsymbol{B}_0$ and to the magnetic field $\boldsymbol{B}_a$ generated by the uniform current i of the central axis. When the ball is rotating, the charges are also subjected, to first order, to the induced current density $\boldsymbol{J}_1 = \sigma(\boldsymbol{E}_1 + \boldsymbol{v} \times \boldsymbol{B}_0)$, its magnetic field $\boldsymbol{B}_1$ and the current $\boldsymbol{J}_2 = \sigma(\boldsymbol{E}_2 + \boldsymbol{v} \times \boldsymbol{B}_a)$ with its associated magnetic field $\boldsymbol{B}_2$. The electric fields $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are produced from the redistribution of the charges within the ball, while σ represents the conductivity of the metallic ball. The aforementioned induced currents and fields represent the result of a perturbative description of the complicated electromagnetic structure within a ball, where only terms of order v/c are retained, while higher-order relativistic terms are neglected.

2. Calculation of currents and magnetic fields

We consider a single ball of radius R where the current is assumed to enter and leave the ball over small spherical caps defined by a half angle $a \neq 0$. This assumption has been correctly introduced by Gruenberg because for $a = 0$ the current density is infinite and the magnetic field tends to infinity in the neighbourhood of the poles ($\theta = 0, \pi$). In this case, it can be easily shown that the series representations of various quantities like the torque and the ohmic power loss diverge as $a \to 0$. If I denotes the current which enters and leaves a ball, the total motor current is $i = N_1 I$ where N_1 is the number of balls per bearing. For a constant angular velocity the total currents and fields are stationary and satisfy the static Maxwell equations

$$\nabla \times \boldsymbol{E} = 0 \qquad \nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0 \tag{1}$$

$$\nabla \times \boldsymbol{B} = \mu \boldsymbol{J} \qquad \nabla \cdot \boldsymbol{B} = 0 \tag{2}$$

where ρ is the charge, ε_0 is the permittivity constant and μ is the magnetic permeability. The total current satisfies the continuity equation

$$\nabla \cdot \boldsymbol{J} = 0 \tag{3}$$

while for moving bodies, the current density is given by Ohm's law

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}). \tag{4}$$

The approximated total fields and currents generated within a ball are written

$$\boldsymbol{E} = \boldsymbol{E}_0 + \boldsymbol{E}_1 + \boldsymbol{E}_2 \qquad \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_a + \boldsymbol{B}_1 + \boldsymbol{B}_2 \qquad \boldsymbol{J} = \boldsymbol{J}_0 + \boldsymbol{J}_1 + \boldsymbol{J}_2 \tag{5}$$

where the terms $\boldsymbol{E}_0$, $\boldsymbol{B}_0$, $\boldsymbol{J}_0$ and $\boldsymbol{B}_a$ exist even when the ball is not rotating. The terms $\boldsymbol{E}_1$, $\boldsymbol{B}_1$ and $\boldsymbol{J}_1$ represent the perturbations introduced, to first order, by the action of the force $-e(\boldsymbol{v} \times \boldsymbol{B}_0)$. The terms $\boldsymbol{E}_2$ and $\boldsymbol{J}_2$ are generated to first order by the effect of the $-e(\nabla \times \boldsymbol{B}_a)$ force. Applying equations (1) and (2) to (5) and appropriately separating the various perturbative terms, we obtain the following system of equations:

$$\nabla \times \boldsymbol{E}_j = 0 \qquad \nabla \cdot \boldsymbol{E}_j = -G_j \qquad j = 0, 1, 2 \tag{6}$$

$$\nabla \times \boldsymbol{B}_j = \mu \boldsymbol{J}_j \qquad \nabla \cdot \boldsymbol{B}_j = 0 \tag{7}$$

$$G_0 = 0 \qquad G_1 = \nabla \cdot (\boldsymbol{v} \times \boldsymbol{B}_0) \qquad G_2 = \nabla \cdot (\boldsymbol{v} \times \boldsymbol{B}_a) \tag{8}$$

$$\boldsymbol{J}_0 = \sigma \boldsymbol{E}_0 \qquad \boldsymbol{J}_1 = \sigma(\boldsymbol{E}_1 + \boldsymbol{v} \times \boldsymbol{B}_0) \qquad \boldsymbol{J}_2 = \sigma(\boldsymbol{E}_2 + \boldsymbol{v} \times \boldsymbol{B}_a). \tag{9}$$

For $j=0$ we obtain the zero-order system of equations while for $j=1,2$ we have the corresponding first-order equations.

However, the total field $\boldsymbol{B}_\mathrm{a}$ is not calculated from the corresponding equations but from the Biot-Savart integral

$$\boldsymbol{B}_\mathrm{a}=\frac{\mu I N_1}{4\pi}\int_{l_0}^{0}\frac{\mathrm{d}\boldsymbol{l}\times(\boldsymbol{r}-\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|^3}. \tag{10}$$

Here the current of the central axis at the point $\boldsymbol{r}_0(l,0,-b)$ generates a magnetic field which is applied at the point $\boldsymbol{r}=(x,y,z)$ of the ball. The length l_0 is the distance between the centres of the two bearings and $\mathrm{d}\boldsymbol{l}=(-\mathrm{d}l,0,0)$.

The required fields and currents are determined as follows. We define $\boldsymbol{E}_j=-\nabla\psi_j$ and solve the corresponding Laplace or Poisson equations of the form

$$\nabla^2\psi_j=G_j. \tag{11}$$

From (9) we obtain the corresponding current densities $\boldsymbol{J}_j$ and finally we determine the induced magnetic fields $\boldsymbol{B}_j$ by solving, instead of (7), the single equations

$$\nabla^2\boldsymbol{B}_j=-\mu\nabla\times\boldsymbol{J}_j. \tag{12}$$

All the differential equations will be solved using the spherical coordinates r, φ and θ.

Starting from the zero-order equations with $j=0$ we observe that the solution of (11), ψ_0, will be independent of φ, with odd symmetry about $\theta=\pi/2$ and bounded in the region $0\leqslant r\leqslant R$, because of the cylindrical symmetry about the z axis. Hence ψ_0 is obtained from the general solution of the Laplace equation

$$\psi_\mathrm{h}=\sum_{n',m'}A_{n',m'}\left(\frac{r}{R}\right)^{n'}P_{n'}^{m'}(\cos\theta)\begin{Bmatrix}\cos m'\varphi\\ \sin m'\varphi\end{Bmatrix} \tag{13}$$

with $m'=0$ and $n'=2n+1$. The constants A_{2n+1} are determined from the boundary condition that the radial component of the current density vector $J_{0r}=-\sigma(\partial\psi_0/\partial r)$ must vanish over the surface of the ball, except at the two spherical caps, where the current enters ($\theta=0$) and leaves ($\theta=\pi$). This is written as

$$I=\mp 2\pi R^2\sigma\int_\mathrm{cap}\left(\frac{\partial\psi_0}{\partial r}\right)_{r=R}\sin\theta\,\mathrm{d}\theta \tag{14}$$

where the signs $-$ and $+$ indicate the upper and lower cap, respectively. Hence we obtain the solution

$$\psi_0=\frac{I}{2\pi\sigma R}\sum_{n=0}^{\infty}\frac{4n+3}{2n+1}\left(\frac{r}{R}\right)^{2n+1}B_n(a)P_{2n+1}(\cos\theta) \tag{15}$$

where

$$B_n(a)=\frac{\cot(a/2)P_{2n+1}^1(\cos\theta)}{2(2n+1)(n+1)}\qquad\lim_{a\to 0}B_n(a)=1. \tag{16}$$

From the first equation in (9) we obtain the components of the current density $\boldsymbol{J}_0$

$$J_{0r}=\frac{-I}{2\pi R^2}\sum_{n=0}^{\infty}(4n+3)B_n(a)\left(\frac{r}{R}\right)^{2n}P_{2n+1}(\cos\theta) \tag{17}$$

$$J_{0\theta}=\frac{I}{2\pi R^2}\sum_{n=0}^{\infty}\frac{4n+3}{2n+1}B_n(a)\left(\frac{r}{R}\right)^{2n}P_{2n+1}^1(\cos\theta) \tag{18}$$

$$J_{0\varphi}=0. \tag{19}$$

Substituting these components into (7) and considering that the lines of $\boldsymbol{B}_0$ are concentric circles around the z axis, we obtain the only two non-vanishing components

$$\frac{1}{r}\left(\cos\theta B_{0\varphi}+\frac{\partial B_{0\varphi}}{\partial\theta}\right)=\mu J_{0r}\qquad -\frac{1}{r}\frac{\partial}{\partial r}(rB_{0\varphi})=\mu J_{0\theta}. \tag{20}$$

Integrating (20) we obtain

$$\boldsymbol{B}_0=-\frac{\mu I}{2\pi R}\sum_{n=0}^{\infty}\frac{4n+3}{2(2n+1)(n+1)}\left(\frac{r}{R}\right)^{2n+1}P^1_{2n+1}(\cos\theta)B_n(a). \tag{21}$$

This expression has been shown by Gruenberg to be reduced to the well known formula

$$|B_0|=\frac{\mu I}{2\pi R\sin\theta}\qquad \text{for } r=R,\qquad a<\theta<\pi-a. \tag{22}$$

The solution of equations (6) for the first-order equations $j=1$ is reduced to the solution of (11) with $G_1=\boldsymbol{\nabla}\cdot(\boldsymbol{v}\times\boldsymbol{B}_0)$ and $\boldsymbol{v}\times\boldsymbol{B}_0=-\hat{\boldsymbol{r}}\Omega rB_0\sin\varphi$. Hence the electric field $\boldsymbol{E}_1$ is determined from the potential ψ_1 which satisfies the Poisson equation

$$\nabla^2\psi_1=\frac{\mu I\Omega\sin\varphi}{2\pi}\sum_{n=0}^{\infty}\frac{(4n+3)(n+2)}{(2n+1)(n+1)}\left(\frac{r}{R}\right)^{2n+1}P^1_{2n+1}(\cos\theta)B_n(a). \tag{23}$$

The solution at the surface of the sphere is subjected to the boundary condition $\hat{\boldsymbol{e}}_r\cdot\boldsymbol{\nabla}\psi_1=(\boldsymbol{v}\times\boldsymbol{B}_0)\cdot\hat{\boldsymbol{e}}_r$ at $r=R$, which takes the form

$$\frac{\partial\psi_1}{\partial r}+\Omega rB_0\sin\varphi=0\qquad \text{at } r=R. \tag{24}$$

The potential ψ_1 is the sum of the solution of the homogeneous equation $\psi_{1\mathrm{h}}$ multiplied by $B_n(a)$ and of the particular solution $\psi_{1\mathrm{p}}$. The function ψ_1 is of the form (13) with $m'=1$, $n'=2n+1$ while the function $\psi_{1\mathrm{p}}$ is obtained following the method of appendix 1. After the application of the condition (24), one obtains

$$\psi_1=\frac{\mu I\Omega R\sin\varphi}{4\pi}\sum_{n=0}^{\infty}\frac{(4n+3)}{(2n+1)(4n+5)}\left[\frac{n+1}{n+2}\left(\frac{r}{R}\right)^2-1\right]\times\left(\frac{r}{R}\right)^{2n+1}R^1_{2n+1}(\cos\theta)B_n(a). \tag{25}$$

Applying (9) and (12), we obtain the corresponding current $\boldsymbol{J}_1$ and the magnetic field $\boldsymbol{B}_1$:

$$J_{1r}=\frac{\mu I\Omega\sigma\sin\varphi}{4\pi}\sum_{n=0}^{\infty}\frac{4n+3}{4n+5}\left[1-\left(\frac{r}{R}\right)^2\right]\left(\frac{r}{R}\right)^{2n}B_n(a)P^1_{2n+1}(\cos\theta) \tag{26}$$

$$J_{1\theta}=\frac{\mu I\Omega\sigma\sin\varphi}{4\pi}\sum_{n=0}^{\infty}\frac{4n+3}{(4n+5)(2n+1)}\left[1-\frac{n+2}{n+1}\left(\frac{r}{R}\right)^2\right]\times\left(\frac{r}{R}\right)^{2n}B_n(a)\frac{\mathrm{d}}{\mathrm{d}\theta}P^1_{2n+1}(\cos\theta) \tag{27}$$

$$J_{1\varphi}=\frac{\mu I\Omega\sigma\cos\varphi}{4\pi\sin\theta}\sum_{n=0}^{\infty}\frac{4n+3}{(4n+5)(2n+1)}\left[1-\frac{n+2}{n+1}\left(\frac{r}{R}\right)^2\right]\times\left(\frac{r}{R}\right)^{2n}B_n(a)P^1_{2n+1}(\cos\theta) \tag{28}$$

$$B_{1r}=0 \tag{29}$$

$$B_{1\theta}=\frac{\mu^2 I\Omega\sigma R}{8\pi}\sum_{n=0}^{\infty}\frac{(4n+3)}{(4n+5)(2n+1)(n+1)}\frac{\cos\varphi}{\sin\theta}\left[1-\left(\frac{r}{R}\right)^2\right]\times\left(\frac{r}{R}\right)^{2n+1}B_n(a)P^1_{2n+1}(\cos\theta) \tag{30}$$

$$B_{1\varphi}=\frac{-\mu^2 I\Omega\sigma R\sin\varphi}{8\pi}\sum_{n=0}^{\infty}\frac{(4n+3)}{(4n+5)(2n+1)(n+1)}\left[1-\left(\frac{r}{R}\right)^2\right]\times\left(\frac{r}{R}\right)^{2n+1}B_n(a)\frac{\mathrm{d}}{\mathrm{d}\theta}P^1_{2n+1}(\cos\theta). \tag{31}$$

The solution of (6) for $j=2$ is reduced to the solution of (11) with $G_2=\nabla\cdot(\boldsymbol{v}\times\boldsymbol{B}_\mathrm{a})$. The integration of (10) leads, in a spherical coordinate system, to the formula

$$\boldsymbol{B}_a=\frac{\mu I N_1}{4\pi D}\left(\frac{x}{|\boldsymbol{r}-\boldsymbol{b}|}-\frac{x-l_0}{|\boldsymbol{r}-\boldsymbol{A}|}\right)(-(b+z)\sin\theta\sin\varphi+y\cos\theta,\; -(b+z)\cos\theta\sin\varphi-y\sin\theta,\; -(b+z)\cos\varphi) \tag{32}$$

where x, y, z are the rectangular coordinates and $D=y^2+(b+z)^2$. The vector $\boldsymbol{b}=(0,0,-b)$ denotes the distance of the centre of the ball from the centre of the rotating axis and the vector $\boldsymbol{A}=(l_0,0,-b)$ has $\varphi=0$ and $\varepsilon=\cos\theta_A=-b/A$. In rectangular coordinates we have

$$\boldsymbol{v}\times\boldsymbol{B}_\mathrm{a}=\frac{i\mu I N_1\Omega b}{4\pi D}\left(\frac{xy}{|\boldsymbol{r}-\boldsymbol{b}|}-\frac{y(x-l_0)}{|\boldsymbol{r}-\boldsymbol{A}|}\right). \tag{33}$$

The electric field $\boldsymbol{E}_2$ can now be determined from a potential ψ_2 which satisfies Poisson's equation

$$\nabla^2\psi_2=\frac{\mu I\Omega N_1 by}{4\pi}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{b}|^3}-\frac{1}{|\boldsymbol{r}-\boldsymbol{A}|^3}\right). \tag{34}$$

The solution ψ_2 is subjected to the boundary condition that the radial component of $\boldsymbol{J}_2$ must vanish over the surface of the ball, so that

$$\int\nabla\psi_2\cdot\boldsymbol{e}_r\,\mathrm{d}S=\int(\boldsymbol{v}\times\boldsymbol{B}_\mathrm{a})\boldsymbol{e}_r\,\mathrm{d}S. \tag{35}$$

Let us write $\psi_2=\psi_{2,b}+\psi_{2,A}$ where $\psi_{2,b}$ and $\psi_{2,A}$ are the particular solutions of the differential equations obtained from the separation of (34), keeping only the first or the second term of the right-hand side of (34), respectively. The series representations of the terms of (34) in terms of the single Legendre polynomials lead to

$$\nabla^2\psi_{2,b}=\frac{\mu I\Omega N_1}{4\pi b}\sum_{n=0}^{\infty}(-1)^n\left(\frac{r}{b}\right)^n P^1_n(\cos\theta)\sin\varphi \tag{36}$$

$$\nabla^2\psi_{2,A}=\frac{-\mu I\Omega N_1 b}{4\pi A^2}\sum_{n=0}^{\infty}\sum_{q=n-1}^{0}(2q+1)\left(\frac{r}{A}\right)^n P_q(\cos\gamma)\sin\theta\sin\varphi \tag{37}$$

where $q=n-2k-1$ and γ is the angle between $\boldsymbol{r}$ and $\boldsymbol{A}$. Using the relation

$$P_q(\cos\gamma)=P_q(\cos\theta)P_q(\varepsilon)+2\sum_{m=1}^{q}\frac{(q-m)!}{(1+m)!}P^m_q(\varepsilon)P^m_q(\cos\theta)\cos m(\varphi_r-\varphi_A) \tag{38}$$

with $\varphi_A = 0$ and $\varepsilon = \cos\theta_A$, we ask for a solution of the form

$$\psi_{2,A} = \frac{-\mu I \Omega N_1}{4\pi} \sum_{n=0}^{\infty} \sum_{q=n-1}^{0} \frac{(2q+1)}{A^{n+2}} \left(P_q(\varepsilon)\psi_{2,A}^{(1)} + 2 \sum_{m=1}^{q} \frac{(q-m)!}{(q+m)!} P_q^m(\varepsilon)\psi_{2,A}^{(2)} \right). \tag{39}$$

Hence (37) is reduced to the solution of the equations

$$\nabla^2 \psi_{2,A}^{(1)} = r^n P_q(\cos\theta)\sin\theta\sin\varphi \qquad \nabla^2 \psi_{2,A}^{(2)} = r^n P_q^m(\cos\theta)\sin\theta\cos m\varphi\sin\varphi. \tag{40}$$

Solving the above equations by applying the methods developed in appendix 1 with $g_1(n) = 1$, $m = 1$ and appendix 2, we determine the potential ψ_2, which is given by

$$\psi_2 = \frac{\mu I \Omega N_1 b}{4\pi} \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2(2n+3)} \left(\frac{r}{b}\right)^{n+2} P_n^1(\cos\theta)\sin\varphi \right.$$
$$- \sum_{q=n-1}^{0} \left(\frac{r}{A}\right)^{n+2} \left(P_q(\varepsilon) Q_{1,2}^{q,1}(\theta)\sin\varphi + \sum_{m=1}^{q} \frac{(q-m)!}{(q+m)!} P_q^m(\varepsilon) \right.$$
$$\left.\left. \times [Q_{1,2}^{q,n+1}(\theta)(\Phi_0 + \tfrac{1}{2}\Phi_1) + Q_{3,4}^{q,m-1}\tfrac{1}{2}\Phi_1] \right) \right] \tag{41}$$

where $Q_{i,j}^{q,m} = c_i P_{q+1}^m - c_j P_{q-1}^m$. The coefficients c_i are defined in appendix 2, $\Phi_0 = \cos m\varphi \sin\varphi$ and $\Phi_1 = \sin(m-1)\varphi$.

The potential ψ_2 satisfies the boundary condition (35). The addition of a solution to the homogeneous equation of the form (13), subjected to the condition (35), would lead to zero constants.

Using $\boldsymbol{E}_2 = -\boldsymbol{\nabla}\psi_2$ and (9), one finally obtains the current density $\boldsymbol{J}_2$

$$J_{2r} = \frac{\mu I \Omega N_1 \sigma}{4\pi} \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1}(n+2)}{2(2n+3)} \left(\frac{r}{b}\right)^{n+1} P_n^1(\cos\theta)\sin\varphi \right.$$
$$+ \frac{b}{A} \sum_{q=n-1}^{0} \left(\frac{r}{A}\right)^{n+1} (n+2) \left(P_q(\varepsilon) Q_{1,2}^{q,1}(\theta)\sin\varphi \right.$$
$$\left.\left. + 2 \sum_{m=1}^{q} \frac{(q-m)!}{(q+m)!} P_q^m(\varepsilon)(Q_{1,2}^{q,m+1}(\theta)\Phi_2 + Q_{3,4}^{q,m-1}(\theta)\tfrac{1}{2}\Phi_1) \right) \right]$$
$$+ (\boldsymbol{v} \times \boldsymbol{B}_{\mathrm{a}})\boldsymbol{e}_r \tag{42}$$

$$J_{2\theta} = \frac{\mu I \Omega N_1 \sigma}{4\pi} \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1}}{2(2n+3)} \left(\frac{r}{b}\right)^{n+1} \sin\theta \frac{\mathrm{d}P_n^1}{\mathrm{d}\cos\theta} \sin\varphi \right.$$
$$- \frac{b}{A} \sum_{q=n-1}^{0} \left(\frac{r}{A}\right)^{n+1} \sin\theta \left(P_q(\varepsilon) Q_{1,2}^{q,1}(\theta)\sin\varphi + 2 \sum_{m=1}^{q} \frac{(q-m)!}{(1+m)!} P_q^m(\varepsilon) \right.$$
$$\left.\left. \times \sin\theta \frac{\mathrm{d}}{\mathrm{d}\cos\theta} (Q_{1,2}^{q,m+1}(\theta)\Phi_2 + Q_{3,4}^{q,m-1}(\theta)\tfrac{1}{2}\Phi_1) \right) \right] + (\boldsymbol{v} \times \boldsymbol{B}_{\mathrm{a}})\boldsymbol{e}_\theta \tag{43}$$

$$J_{2\varphi} = \frac{\mu I \Omega N_1 \sigma}{4\pi} \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1}}{2(2n+3)} \left(\frac{r}{b}\right)^{n+1} P_n^1(\cos\theta) \frac{\cos\varphi}{\sin\theta} \right.$$
$$+ \frac{b}{A} \sum_{q=n-1}^{0} \left(\frac{r}{A}\right)^{n+1} \left(P_q(\varepsilon) Q_{1,2}^{q,1}(\theta) \frac{\cos\varphi}{\sin\theta} \right.$$
$$\left.\left. + 2 \sum_{m=1}^{q} \frac{(q-m)!}{(q+m)!} \frac{P_q^m(\varepsilon)}{\sin\theta} (Q_{1,2}^{q,m+1}\Phi_3 + Q_{3,4}^{q,m-1}(\theta)\tfrac{1}{2}\Phi_4) \right) \right] + (\boldsymbol{v} \times \boldsymbol{B}_{\mathrm{a}})\boldsymbol{e}_\varphi \tag{44}$$

where $\Phi_2 = \Phi_0 + \Phi_1/2$, $\Phi_3 = -(m+1)(\sin m\varphi \sin\varphi - \frac{1}{2}\cos(m-1)\varphi)$ and $\Phi_4 = (m-1)\cos(m-1)\varphi$.

Substituting the leading terms of $\boldsymbol{J}_2$ into (12), we determine the corresponding terms of the induced magnetic field $\boldsymbol{B}_2$. However, the estimated leading terms indicate that the contribution of this field to the torque is negligible.

3. Torque, power and ohmic power loss

The total torque applied on the sphere is given by the volume integral

$$T=\int (\boldsymbol{J}\times\boldsymbol{B})\times\boldsymbol{r}\cdot\boldsymbol{i}\,\mathrm{d}^3r. \qquad (45)$$

The force $\boldsymbol{J}\times\boldsymbol{B}$ is the result up to first order of the current density $\boldsymbol{J}=\boldsymbol{J}_0+\boldsymbol{J}_1+\boldsymbol{J}_2$ and of the field $\boldsymbol{B}=\boldsymbol{B}_0+\boldsymbol{B}_1+\boldsymbol{B}_a$. The terms $\boldsymbol{J}_i\times\boldsymbol{B}_i$, $(i=0,1)$ do not contribute to the torque because of the Newton's third law. After some algebra, the terms $F_0=\boldsymbol{J}_0\times\boldsymbol{B}_1+\boldsymbol{J}_1\times\boldsymbol{B}_0$ and $\boldsymbol{J}_0\times\boldsymbol{B}_a$ are also integrated to zero. Our result for the force F_0 contradicts Gruenberg's result due to his algebraic errors with respect to the coefficients of $\boldsymbol{J}_1$, $\boldsymbol{B}_1$ and the integration.

Weenink agrees with our result, but he has not applied the boundary condition (14) correctly so that the term $B_n(a)$ does not appear in his expression for the currents and fields. However, $B_n(a)$ is given by the relation (16) and is of vital importance for the convergence of the series expressions for the torque, the total power and the ohmic power loss, which will be evaluated below. Of course Gruenberg and Weenink had not taken into account the forces acting on the ball which were generated by the induced currents and fields caused by the current of the central axis.

The integration of (45) is achieved when the field $\boldsymbol{B}_a$ and the current density $\boldsymbol{J}_2$, given by the equations (32) and (42)-(44), respectively, are written in the form of series, via the relations

$$\begin{aligned}\frac{1}{D}&=\frac{1}{(b+z)^2}\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\left(\frac{y}{b+z}\right)^{2(\nu-1)}\\ \frac{1}{(b+z)^n}&=\frac{1}{b^n}\sum_{\lambda=0}^{\infty}\varepsilon_\lambda(n)\left(\frac{r}{b}\right)^{\lambda}\cos^\lambda\theta\end{aligned} \qquad (46)$$

where $\varepsilon_\lambda(n)=\Pi_{i=0}^{\lambda-1}(-n+i)$.

After tedious calculations we have found the following formulae for the non-vanishing terms which contribute to the torque of the ball:

$$\begin{aligned}T_1=\int(\boldsymbol{J}_2\times\boldsymbol{B}_0)\times\boldsymbol{r}\cdot\boldsymbol{i}\,\mathrm{d}r^3&=\frac{\mu^2I^2\Omega N_1\sigma R^3}{4\pi}\sum_{n,n'=0}^{\infty}\frac{(4n'+3)B_{n'}(a)}{(2n'+1)(n'+1)}\\ &\times\left\{\frac{(-1)^{n+1}(n+2)(n+1)n}{2(2n+3)(2n+1)}\left(\frac{R}{b}\right)^n\frac{c_5}{b}\delta_{2n'+1,n}+\sum_{q=n-1}^{0}\frac{(n+2)}{4\pi}\frac{b}{A^2}\left(\frac{R}{A}\right)^n\right.\\ &\times c_5\left(P_q(\varepsilon)Z_{1,2}(2n'+1,1;q,1)\pi+2\sum_{m=0}^{q}\frac{(q-m)!}{(q+m)!}P_q^m(\varepsilon)W_3(n',q,m)\right)\\ &+\sum_{\substack{\lambda=0\\ \nu=1}}^{\infty}\frac{(-1)^{\nu-1}\varepsilon_\lambda(2\nu)}{2}\left(\frac{R}{b}\right)^{2\nu+\lambda-1}\\ &\times\left[R^nG_nW_1(0,2;n,n',\nu,\lambda)C_6+2\left(\frac{R}{A}\right)^n\sum_{m=1}^{q}\frac{(q-m)!}{(q+m)!}P_q^m(\varepsilon)\right.\\ &\left.\left.\times\left(\frac{c_6}{A}W_1(m,2;n,n',\nu,\lambda)+\frac{l_0c_7}{R}W_1(m,l;n,n',v,\lambda)\right)\right]\right\}\end{aligned} \qquad (47)$$

$$T_2 = \int (\boldsymbol{J}_1 \times \boldsymbol{B}_a) \times \boldsymbol{r} \cdot \boldsymbol{i}\, \mathrm{d}r^3 = \frac{\mu^2 I^2 \Omega N_1 \sigma R^3}{8\pi} \sum_{\substack{n,n',\lambda=0\\ \nu=1}}^{\infty} (-1)^{\nu-1} \varepsilon_\lambda (2\nu) \left(\frac{4n'+3}{4n'+5}\right) B_{n'}(a)$$

$$\times \left(\frac{R}{b}\right)^{2\nu+\lambda-1} \Bigg(G_n M(0,2,0,0; n, n', \nu, \lambda) - 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} \frac{P_n^m(\varepsilon)}{A}$$

$$\times (M(m,2,m,0; n,n',\nu,\lambda) - l_0 M(m,l,m,m; n,n',\nu,\lambda))\Bigg) \tag{48}$$

where

$$q = 2n - k + 1 \qquad c_5 = (2n' + n + 5)^{-1} \qquad c_6 = (2n' + n + 2\nu + \lambda + 4)^{-1},$$

$$c_7 = (2n' + n + 2\nu + \lambda + 3)^{-1} \qquad \text{and} \qquad G_n = \frac{(-1)^n}{b^{n+1}} - \frac{P_n(\varepsilon)}{A^{n+1}}$$

$$W_0(\nu, m; i, j; n, n', \lambda)$$

$$= C(2\nu, m, i)[S_2(n, j; 2n'+1, 1; \lambda+1, 2\nu+i-1)$$

$$- S_1(n, m; 2n'+1, 1; \lambda, 2\nu+i-3)]$$

$$W_1(m, i; n, n', \nu, \lambda) = C(2\nu, m, i) S_1(n, m; 2n'+1, 1; \lambda, 2\nu+i-1)$$

$$W_3(n', q, m) = Z_{1,2}(2n'+1, 1; q, m+1)(2\pi C(2, m, 0) + \tfrac{1}{2}\pi\delta(m-2))$$

$$+ Z_{3,4}(2n'+1, 1; q, m-1)\tfrac{1}{2}\pi\delta(m-2)$$

$$Z_{i,j}(n', m'; q, m) = \int_{-1}^{1} P_{n'}^{m'} Q_{i,j}^{q,m}\, \mathrm{d}\cos\theta$$

$$= c_i S_1(n', m'; q+1, m; 0, 0) - c_j S_1(n', m'; q-1, m; 0, 0)$$

$$M_1(j, i, k, e; n, n', \nu, \lambda) = L_1(d, i) W_1(j, i; n, n', \nu, \lambda) + \frac{L_2(d, i)}{2n'+1} W_0(\nu, k; i, e; n, n', \lambda)$$

$$L_1(d, \sigma) = \frac{2R^{n+\sigma+2}}{(d+\sigma)(d+\sigma+2)} \qquad L_2(d, \sigma) = \tfrac{1}{2} L_1(d, \sigma)\left[(d+\sigma+2) - \left(\frac{n'+2}{n+1}\right)(d+\sigma)\right] \tag{49}$$

where $d = 2n' + 2\nu + n + \lambda$. The calculated integrals

$$C(\nu, m, \sigma) = \frac{1}{2\pi} \int_0^{2\pi} \sin^\nu\varphi \cos m\varphi \cos^\sigma\varphi\, \mathrm{d}\varphi \tag{50}$$

$$S_1(n, m; n', m'; p, q) = \int_{-1}^{1} \sin^q\theta \cos^P\theta P_{n'}^{m'} P_n^m\, \mathrm{d}\cos\theta \tag{51}$$

$$S_2(n, m; n', m'; p, q) = \int_{-1}^{1} \sin^q\theta \cos^P\theta \frac{\mathrm{d}P_{n'}^{m'}}{\mathrm{d}\cos\theta} P_n^m\, \mathrm{d}\cos\theta \tag{52}$$

are given by (A3.1), (A3.4) and (A3.5) in appendix 3. Since the force $\boldsymbol{J}_2 \times \boldsymbol{B}_a$ is very weak, we have considered only the leading contributions to the torque, neglecting terms of order A^{-n} with $n \geq 1$. This term of the torque is given by

$$T_3 = \int (\boldsymbol{J}_2 \times \boldsymbol{B}_a) \times \boldsymbol{r} \cdot \boldsymbol{i} \, \mathrm{d}r^3$$

$$= \frac{\mu^2 I^2 \Omega N_1^2 \sigma R^3}{8\pi} \sum_{\substack{n',n,\lambda=0 \\ \nu=1}}^{\infty} \frac{(-1)^{n+n'+\nu+1}}{2(2n'+3)} \left(\frac{R}{b}\right)^{2\nu+n+n'+\lambda+1} c_8 C(2\nu, 0, 2)$$

$$\times [(S_1(n, 0; n', 1; \lambda+2, 2\nu-1) - S_1(n, 0; n', 1; \lambda, 2\nu-1))(n'+2)$$

$$- S_2(n, 0; n', 1; \lambda+1, 2\nu+1) + S_1(n, 0; n', 1; \lambda, 2\nu-1)] \qquad (53)$$

where $c_8 = (2\nu + \lambda + n + n' + 4)^{-1}$.

The contribution of the force $\boldsymbol{J}_2 \times \boldsymbol{B}_1$ in the torque is not presented, because the leading terms are of the order 10^{-5}.

The torque $T = T_1 + T_2 + T_3$ is positive and tends to increase the speed until equilibrium is established with frictional and windage torques.

The total power P and the ohmic power loss P_L are given by the relations

$$P = NT\Omega \qquad P_L = \frac{N}{\sigma} \int |\boldsymbol{J}_0 + \boldsymbol{J}_1 + \boldsymbol{J}_2| \, \mathrm{d}r^3 \qquad (54)$$

where $N = N_1 N_2$; N_1 is the number of balls per bearing and N_2 is the number of bearings. Taking into account the dependence of $\boldsymbol{J}_i$ ($i = 0, 1, 2$) on the angle φ, omitting terms of the order A^{-n} with $n \geq 1$ and neglecting the terms J_1^2 and J_2^2 which are of the order $(v/c)^2$, we obtain the relation

$$P_L = \frac{N}{\sigma} \int |J_0| \, \mathrm{d}^3 r = \frac{NI^2}{\sigma \pi R} \sum_{n=0}^{\infty} \frac{4n+3}{2n+1} B_n^2(a). \qquad (55)$$

The series in (55) as well as the series respesentations of the torque in (47), (48) clearly diverge in the limit $a \to 0$ since $B_n(0) = 1$. To make the results useful, we adopt Gruenberg's analysis, where for very small but finite angles a, the formula (16) for $B_n(a)$ is approximated by

$$B_n(a) \simeq \frac{4}{a} \frac{J_1[(4n+3)(a/2)]}{4n+3} \qquad (56)$$

where J_1 is the Bessel function of order one. For fixed a, J_1 tends to zero rapidly after $4n+3$ surpasses $2/a$ and the series in (47), (48) and (55) converge. Substituting (56) into (55) and evaluating the resulting sum numerically we arrive at the approximated formula

$$P_L = \frac{0.541 NI^2}{R\sigma a}. \qquad (57)$$

4. Motor characteristics and comparison with experiment

In order to compare our theoretical results with the corresponding experimental ones, it is necessary to express the required quantities in terms which are used for the description of the motor characteristics. So the system of bearings is subjected to voltage V and total current $i = IN_1$ while the angular velocity of the shaft is given by

$$\omega = \frac{R}{(b-R)} \Omega = k_0 \Omega. \qquad (58)$$

In terms of these variables the total power developed by the motor P, the torque and the ohmic loss are

$$P = k\omega^2 i^2 \qquad T = kk_0\omega i^2 \qquad P_L = Re\, i^2 \tag{59}$$

where Re is the resistance and k is a constant depending on the motor characteristics.

The torque is written $T = \Sigma_{i=1}^3 T_i = \mu^{-2} I^2 \Omega\sigma R^3 tN$ where $t = \Sigma_{i=1}^3 t_i$. The terms t_i incorporate the corresponding series of the terms T_i. Calculating t_i from (47), (48) and (53), we have neglected terms of order $(R/b^2)^n$ with $n > 1$ and A^{-n}, with $n \geqslant 1$, having estimated values smaller than 10^{-5}. The latter case was expected by the fact that the field $\boldsymbol{B}_a$ is mainly generated by the current of the central axis in the vicinity of the ball. Hence terms of the order A^{-n} in the vectors $\boldsymbol{B}_a$, $\boldsymbol{J}_2$ and the generated forces are of minor significance. Under the above assumptions and calculating all the terms of order R/b^2, we obtained the values $t_1 = 0.151\,37\ N_1$, $t_2 = 0.005\,66\ N_1$, $t_3 = 3.857 \times 10^{-5} N_1^2$ with total $t = 1.1011$.

This means that the main contribution to the torque is caused by the force $\boldsymbol{J}_2 \times \boldsymbol{B}_0$ and is followed by the contributions due to the forces $\boldsymbol{J}_1 \times \boldsymbol{B}_a$ and $\boldsymbol{J}_2 \times \boldsymbol{B}_a$, as was expected. The comparison of (59) with our aforementioned formulae leads to the relations

$$k = \frac{t(b-R)^2 R\mu^{-2}\sigma N_2}{N_1} \qquad Re = \frac{0.541}{R\sigma Na}. \tag{60}$$

The overall power balance leads to the equation $Vi = P + P_L$ from which we obtain $V = Em + Re \times i$, where $Em = k\omega^2 i$ is the back EMF. Defining

$$\omega_0 = \left(\frac{Re}{3k}\right)^{1/2} = \frac{0.42426}{(b-R)R\mu\sigma(at)^{1/2}} \tag{61}$$

we obtain for the torque, via equation (59), the relation

$$T = \frac{kk_0\omega V^2}{(Re + k\omega^2)^2} = \frac{3k_0}{\omega_0\, Re} \frac{V^2(\omega/\omega_0)}{[3 + (\omega/\omega_0)^2]^2}. \tag{62}$$

The torque which does not depend on the number of bearings becomes zero for $\omega = 0$ and takes its maximum value for $\omega = \omega_0$. The efficiency of the motor is defined by

$$E_f = \frac{P}{P + P_L} = \frac{(\omega/\omega_0)^2}{3 + (\omega/\omega_0)^2}. \tag{63}$$

For small bearings $(\omega/\omega_0)^2 \ll 1$ we have $P \ll P_L$ and $V \simeq Re \times i$.

In the case of large bearings ω_0 may be small enough that stable operation at speeds in excess of ω_0 is feasible. However, for small bearings ω_0 is large and friction and windage losses will lead to angular velocities ω below ω_0. Considering that the torque and the total power increase with the size of the bearing while the ohmic power loss decreases, we conclude that the efficiency E_f tends to increase with the size of the bearing.

The validity of our theory for the explanation of the ball bearing motor will be tested by comparing our theoretical predictions with the corresponding results obtained by us (Moyssides and Hatzikonstantinou 1989) during an experiment for the study of the motor characteristics. In this experiment two bearings with inner diameter 9 mm were used ($N_2 = 2$) in series. Each bearing had $N_1 = 7$ completely spherical balls with radius $R = 2$ mm while the distance of the balls' centre from the centre of the bearing was $b = 11$ mm. Using bearings made of chromium steels and currents varying from $i_{min} = 43.5$ A to $i_{max} = 70.146$ A, we measured resistance $Re(\min i) = 0.0618$ ohm to $Re(\max i) = 0.0383$ ohm, respectively. From extended measurements we estimated a contact angle $a = 7 \times 10^{-5}$ rad. The small size of a enables the flux density to reach saturation levels near the contact points. Considering that for the values of our currents the relative permeability is about $\mu_r = \mu/\mu_0 = 600$ with $\mu_0 = 4\pi \times 10^{-7}$, we estimate from (22), with $\theta = \pi/2$, a magnetic field of the order 0.62–1 T. The magnetic field generated by the central axis (32) is of the order 0.3 T in the centre of the ball. It is known that there is no starting torque unless there is some residual magnetism left in the balls from a previous run. In our experiment using $i_{min} = 43.5$ A and $i_{max} = 70.146$ A we have measured angular velocities $\omega_{min} = 93.18$ rad 5^{-1} and $\omega_{max} = 245.48$ rad s^{-1}, respectively.

The theoretical predictions were estimated using the above parametric values for N_1, N_2, R, b, μ and a and the calculated $t = 1.1011$. The constants k and σ were calculated from (60) using the experimentally measured resistance $Re = 0.05$.

In table 1 we present the theoretical predictions via (60)–(63) and the experimental results for comparison. In both cases the indices min and max of the values $i_{min} = 43.5$ A and $i_{max} = 70.146$ A. It can be easily seen that the theoretical predictions are in excellent agreement with the corresponding experimental results.

Table 1. Theoretical and experimental values of the ball bearing motor characteristics.

Theoretical	Experimental
$\sigma = 5.5204 \times 10^6$ ohm m^{-1} (from (60) with $Re_{exp} = 0.05$, from Moyssides *et al* 1989)	$Re(\min i) = 0.0618 \pm 0013$ ohm
	$Re(\max i) = 0.03831 \pm 0.0008$ ohm
$k = 0.79972 \times 10^{-7}$	$k = (0.80398 \pm 0.304) \times 10^{-7}$ Ws^2 rad^{-2} A^{-2}
$T_{min} = 0.313 \times 10^{-2}$	$T_{min} = (0.315 \pm 0.12) \times 10^{-2}$ Ws rad^{-1}
$T_{max} = 2.146 \times 10^{-2}$	$T_{max} = (2.158 \pm 0.82) \times 10^{-2}$ Ws rad^{-1}
$P_{min} = 1.313$ W	$P_{min} = 1.321 \pm 0.499$ W
$P_{max} = 23.712$ W	$P_{max} = 23.839 \pm 9.054$ W
$P_{L_{min}} = 116.915$ W	$P_{L_{min}} = 116.92 \pm 2.45$ W
$P_{L_{max}} = 188.531$ W	$P_{L_{max}} = 188.53 \pm 4.02$ W
$E_{f_{min}} = 0.0111$	$E_{f_{min}} = 0.011 \pm 0.004$
$E_{f_{max}} = 0.1117$	$E_{f_{max}} = 0.115 \pm 0.045$
$\omega_0(\min i) = 506.1$ rad s^{-1}	
$\omega_0(\max i) = 398.6$ rad s^{-1}	

5. Discussion and conclusions

In this work we have proved that the ball bearing motor cannot be explained by considering only the interactions of the generated currents and associated magnetic fields inside the spherical balls and neglecting the current of the central axis and its associated effects. This is due to the fact that, up to second order, these electromagnetic fields have a structure which yields a zero torque. This result has also been confirmed by Weenink (1981). However, we estimated that higher-order terms ($n \geq 2$) of the electromagnetic fields are of the order $(v/c)^n$, leading to negligible internal excited forces and torque.

For the explanation of this motor behaviour we have developed the aforementioned theory, where for the first time we have taken into account the interactions of the magnetic field generated by the uniform current of the central axis and its associated induced current $\boldsymbol{J}_2$ with the above electromagnetic fields. From our perturbative description, neglecting higher-order relativistic terms, we have found that the main contribution to the torque is caused by the forces $\boldsymbol{J}_2\times\boldsymbol{B}_0$, $\boldsymbol{J}_1\times\boldsymbol{B}_a$ and $\boldsymbol{J}_2\times\boldsymbol{B}_a$.

Calculating the torque, we have neglected from the corresponding series terms of order $(R/b)^n$ with $n>1$ and $A^{-n}=(b^2+l_0^2)^{-n/2}$ with $n\geqslant 1$. Hence the number of perturbative terms required for a given accuracy depends on the relative dimensions of the motor. In practice there are not large limits for variation of the parameter R/b^2. However, as the distance l_0 between the two bearings decreases, the contribution of the higher-order terms A^{-n} to the torque increases.

The excellent agreement between the theoretical and experimental results confirms this theory.

Appendix 1

We wish to solve differential equations of the form

$$\nabla^2\psi = g_1(n)r^nP_n^m(\cos\theta)\sin\sqrt{m}\Phi \tag{A1.1}$$

where $g_1(b)$ depends only on n.

The solution of (A1.1) in spherical coordinates takes the form

$$\psi = cg_1(n)r^{n+2}P_n^m(\cos\theta)\sin\sqrt{m}\varphi \tag{A1.2}$$

where c is a constant. Substituting (A1.2) into (A1.1), we obtain the equation

$$\left(L+g_2(n)-\frac{m}{\sin^2\theta}\right)cP_n^m = g_1(n)P_n^m \qquad \text{with } L=\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta} \tag{A1.3}$$

and $g_2(n)=(n+2)(n+3)$. Solving (A1.3) we find $c=g_1(n)/(g_2(n)-n(n+1))$, which determines the solution (A1.2).

Appendix 2

We wish to solve differential equations of the form

$$\nabla^2\psi = r^nP_q^m(\cos\theta)\cos\varphi\sin\varphi\sin\theta. \tag{A2.1}$$

We consider a solution of the form $\psi=r^{n+2}(\theta_0(\theta)\Phi_0+\theta_1(\theta)\Phi_1)$ where $\Phi_0=\cos m\varphi\sin\varphi$ and $\Phi_1=\sin(m-1)\Phi$. Then from (A2.1) we arrive at the coupled differential equations

$$\left(L+g_2(n)-\frac{(m+1)^2}{\sin^2\theta}\right)\theta_0\Phi_0=\frac{1}{(2q+1)}(P_{q+1}^{m+1}-P_{q-1}^{m+1})\Phi_0 \tag{A2.2}$$

$$\left(L+g_2(n)-\frac{(m-1)^2}{\sin^2\theta}\right)\theta_1\Phi_1=-\frac{2m\Phi_1\theta_0}{\sin^2\theta} \tag{A2.3}$$

where $g_2(n)=(n+2)(n+3)$.

From (A2.2), applying the method of appendix 1, we obtain

$$\theta_0(\theta)=\frac{1}{2q+1}\left(c_1P_{q+1}^{m+1}-c_2P_{q-1}^{m+1}\right) \tag{A2.4}$$

where $c_1=(g_2(n)-(q+1)(q+2))^{-1}$ and $c_2=(g_2(n)-q(q-1))^{-1}$. Putting $\theta_1(\theta)=\frac{1}{2}\theta_0(\theta)+\theta_2(\theta)$, equation (A2.3) takes the form

$$\frac{1}{2}\left(L+g_2(n)-\frac{(m+1)^2}{\sin^2\theta}\right)\theta_0\Phi_1+\left(L+g_2(n)-\frac{(m-1)^2}{\sin^2\theta}\right)\theta_2\Phi_1=0 \tag{A2.5}$$

or

$$\left(L+g_2(n)-\frac{(m-1)^2}{\sin^2\theta}\right)\theta_2\Phi_1=-\frac{1}{2}P_q^m\sin\theta\Phi_1. \tag{A2.6}$$

Substituting into (A2.6) the expression

$$P_q^m=\frac{1}{2q+1}[(q+m)(q+m-1)P_{q-1}^{m-1}-(q-m+1)(q-m+2)P_{q+1}^{m-1}] \tag{A2.7}$$

and again applying the method of appendix 1, we obtain

$$\begin{gathered}\theta_2(\theta)=\frac{1}{2(2q+1)}(c_3P_{q+1}^{m-1}-c_4P_{q-1}^{m-1})\\ c_3=c_1(q-m+1)(q-m+2)\qquad c_4=c_2(q+m)(q+m-1).\end{gathered} \tag{A2.8}$$

Hence the solution of (A2.1) takes the form

$$\psi=r^{n+2}[\theta_0(\Phi_0+\tfrac{1}{2}\Phi_1)+\theta_2\Phi_1]. \tag{A2.9}$$

Appendix 3

(a) The determination of torque is achieved by integrating over the angles φ and θ. The integrals over φ are calculated as follows:

$$\begin{aligned}C(\nu,m,\sigma)&=\frac{1}{2\pi}\int_0^{2\pi}\sin^\nu\varphi\cos m\varphi\cos^\sigma\varphi\,d\varphi\\ &=\sum_{s=0}^{E(m/2)}\binom{m}{25}\frac{(-1)^s}{2\pi}\int_0^{2\pi}\sin^\alpha\varphi\cos^\beta\varphi\,d\varphi\\ &=\sum_{s=0}^{E(m/2)}\binom{m}{25}\frac{(-1)^s(\alpha-1)!!(\beta-1)!!}{(\alpha+\beta)(\alpha+\beta-2)\ldots(\beta+2)2^{\beta/2}(\beta/2)!}\end{aligned} \tag{A3.1}$$

where $E(x)$ means integer value of x, $\alpha=\nu+25$ and $\beta=m+\sigma+25$. The result (A3.1) is correct only when ν and $m+\sigma$ are even numbers, otherwise $C(\nu,m,\sigma)=0$
(b) The integrals over θ have the forms

$$S_1(n,m;n',m';p,q)=\int_0^\pi\sin^q\theta\cos^p\theta P_{n'}^{m'}P_n^m\,d\cos\theta \tag{A3.2}$$

$$S_2(n,m;n',m';p,q)=\int_0^\pi\sin^q\theta\cos^p\theta\frac{dP_{n'}^{m'}}{d\cos\theta}P_n^m\,d\cos\theta. \tag{A3.3}$$

Now we substitute the Legendre polynomials and their derivatives with their corresponding series representations in terms of hypergeometric functions. Hence, the equations (A3.2) and (A3.3) take the form

$$S_1(n, m; n', m'; p, q) = D_0 \sum_{\sigma,\sigma'=0}^{\infty} D_1(\sigma, \sigma') I(p, q/2, q/2+\sigma+\sigma') \tag{A3.4}$$

$$S_2(n, m; n', m'; p, q) = -\sigma D_0 \sum_{\sigma,\sigma'=0}^{\infty} D_1(\sigma, \sigma') I(p, q/2, q/2+\sigma+\sigma'-1) \tag{A3.5}$$

where

$$\begin{aligned} D_0 &= \frac{\Gamma(n+m+1)\Gamma(n'+m'+1)}{2^{m+m'} m!\, m'!\, \Gamma(n-m+1)\Gamma(n'-m'+1)} \\ D_1(\sigma, \sigma') &= \frac{(m-n)_\sigma (m+n+1)_\sigma (m'-n')_{\sigma'} (m'+n'+1)_{\sigma'}}{2^{\sigma+\sigma'} \sigma!\, \sigma'!\, (m+1)_{\sigma'} (m+1)_{\sigma'}} \end{aligned} \tag{A3.6}$$

with $(\alpha)_\sigma = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+\sigma-1)$ and $(\alpha)_0 = 1$. $\Gamma(n)$ represents the gamma function while the integral I with $x = \cos\theta$ is given by

$$\begin{aligned} I(p, q/2, q/2+\sigma+\sigma') &= \int_{-1}^{1} x^p (1+x)^{q/2} (1-x)^{q/2+\sigma+\sigma'}\, dx = B(q/2+\sigma+\sigma'+1, p+1) \\ &\quad \times F(-q/2, p+1; q/2+\sigma+\sigma'+p+2; -1) + (-1)^p B(q/2+1, p+1) \\ &\quad \times F(-q/2-\sigma-\sigma', p+1; q/2+p+2; -1) \end{aligned} \tag{A3.7}$$

where $F(\alpha, \beta; c; x)$ and $B(m, n) = \Gamma(m)\Gamma(n)/\Gamma(m+n)$ are the hypergeometric and beta functions, respectively.

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